Reply to Comment 'On large-N expansion'

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REPLY

Reply to Comment 'On large-N expansion'

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Abstract
Fernandez comments [1] on our pseudo-perturbative shifted-ℓ expansion technique [2, 3] is either unfounded or ambiguous.

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In his comment [1] on our pseudo-perturbation shifted-ℓ expansion technique (PSLET) [2, 3], Fernandez strived to prove that (I) PSLET is just a version of the shifted large-N expansion technique SLNT, (II) it is not true that PSLET enables one to obtain more perturbation corrections than SLNT and (III) it seems that SLNT (and, consequently, also PSLET) is divergent.

We explain below why we believe criticisms (I) and (II) to be unfounded and criticism (III) to be ambiguous.

- Our statement "the difficulty of calculating higher-order corrections in SLNT through Rayleigh–Schrõdinger perturbation theory (RSPT) results in a loss of accuracy" is clear and need not be misleading. We refer to the comprehensive, historical account in the work of Imbo *et al* [4], indicating the actual novelty of SLNT (which could handle, via RSPT, only the first four terms of the energy series). Fernandez and co-workers (in [6–8] of [1]) have used the hypervirial perturbation method (HPM) to calculate higher-order corrections in SLNT. Therefore, one would call their method HPM–SLNT or, at least, *Modified* SLNT (as they themselves named it) and not SLNT.
- We did not claim that PSLET is completely different from SLNT [4] (cf our comment following equation (31) in [3]). At the top of p 3063 in [5], we commented on the higher accuracy of the Fernandez HPM–SLNT method (although we had reservations about the order-dependent shift approach to the Klein–Gordon and Dirac equations).
- Fernandez derived relations between a and β, $\bar{k}$ and $\bar{l}$, ... etc. However, that work simply illustrates part of the message which we tried to deliver to readers, i.e. SLNT is not an expansion in large-N but, in effect, an expansion in large-ℓ (cf Bender *et al* [6]); hence we preferred the abbreviation PSLET.

Table 1. The sum of the first $20y$ terms of the energy series, E_{20}, and the corresponding Padé approximants.

State	E_{20}	Stability starts from	Padé	Stability starts from
4s	−0.011 638	E_{12}	−0.011 638	E[3,3]
6s	−0.006 795	E_{12}	−0.006 7958	E[7,8]
7s	−0.005 443	E_{12}	−0.005 4438	E[7,8]
9s	−0.003 721	E_{13}	−0.003 722	E[7,8]
11s	−0.002 705	E_{14}	−0.002 7068	E[8,8]

Table 2. Energies for $\alpha = 10$, and wavefunctions with ten nodes at $\ell = 1, 3, 5, 15$.

$-E_M$	$\ell = 1$	$\ell = 3$	$\ell = 5$	$\ell = 15$
$-E_0$	0.002 83	0.002 198	0.001 7446	0.000 710 89
$-E_1$	0.002 83	0.002 198	0.001 7446	0.000 710 89
$-E_2$	0.002 67	0.002 118	0.001 7034	0.000 707 89
$-E_3$	0.002 56	0.002 070	0.001 6809	0.000 706 77
$-E_4$	0.002 50	0.002 046	0.001 6699	0.000 706 36
$-E_5$	0.002 48	0.002 034	0.001 6649	0.000 706 22
$-E_6$	0.002 47	0.002 030	0.001 6629	0.000 706 17
$-E_7$	0.002 47	0.002 028	0.001 6621	0.000 706 16
$-E_8$	0.002 47	0.002 028	0.001 6619	0.000 706 15
⋮	0.002 47	0.002 028	0.001 6619	0.000 706 15
$-E_{20}$	0.002 47	0.002 028	0.001 6619	0.000 706 15

- It is true, of course, that our conclusions in [3] about numerical accuracy referred to calculations for state wavefunctions with at most one node. However, the comment by Fernandez that we are unable to apply our method to wavefunctions with more than one node is unjustified, since we have, in fact, given results for such functions in the tables of [2, 7, 8]. Below we also report PSLET results for the truncated Coulomb potential $V(r) = -1/(r + \alpha)$ with $\alpha = 10$ for wavefunctions with several nodes. We show the sum of the first twenty terms of the energy series, E_{20}, and list the corresponding Padé approximants. The orders at which the energy series and Padé approximants stabilize are shown in table 1.
- Fernandez is unjustified in asserting that PSLET is based on logarithmic perturbation theory (LPT) (cf appendix A in [4] and the references cited therein on LPT). PSLET is simply an algebraic recursion method which leads to exactly solvable recursion relations (based on the uniqueness of power series representations, cf [9]).
- It is not universally true that HPM–SLNT and consequently PSLET are divergent. Both techniques are based on asymptotic series expansions and one would expect to get asymptotically divergent or asymptotically convergent results (cf our analysis in [7, 10, 11]). To illustrate this statement with some persuasive evidence we consider the truncated Coulomb potential with $\alpha = 10$, for wavefunctions with ten nodes at $\ell = 1, 3, 5, 15$ (see table 2).

Obviously, the trends of convergence are very well marked. In general, the energy series of SLNT, HPM–SLNT and PSLET are oscillatory (a signal of, at least, asymptotic

convergence) and one would, as a remedy, use an order-dependent shift (as in HPM–SLNT) or Padé approximants (as in PSLET) to obtain results with satisfactory accuracy.

We agree with Fernandez about the unfavourable case ($\alpha = 0.1, \ell = \nu = 0$). Here the energy series appears to be asymptotically divergent. However, this should be attributed mainly to the nature of the truncated Coulomb potential and to the irrational value of α. One should notice that this particular potential gives contributions to the higher-order corrections of the energy series through its non-vanishing higher-order derivatives. This will lead to accumulated rounding-off errors which, in turn, can yield unreliable results from the higher-order corrections.

We believe that the points made above have satisfactorly answered the criticisms (I) to (III) of Fernandez.

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